Exercise 2.3.5

(Dominance of the fittest) Suppose X and Y are two species that reproduce exponentially fast: $\dot{X} = aX$ and $\dot{Y} = bY$, respectively, with initial conditions $X_0, Y_0 > 0$ and growth rates $a > b > 0$. Here X is "fitter" than Y in the sense that it reproduces faster, as reflected by the inequality $a > b$. So we'd expect X to keep increasing its share of the total population $X + Y$ as $t \to \infty$. The goal of this exercise is to demonstrate this intuitive result, first analytically and then geometrically.

a) Let $x(t) = X(t)/[X(t) + Y(t)]$ denote X's share of the total population. By solving for $X(t)$ and $Y(t)$, show that $x(t)$ increases monotonically and approaches 1 as $t \to \infty$.

b) Alternatively, we can arrive at the same conclusions by deriving a differential equation for $x(t)$. To do so, take the time derivative of $x(t) = X(t)/[X(t) + Y(t)]$ using the quotient and chain rules. Then substitute for $\dot{X}$ and $\dot{Y}$ and thereby show that $x(t)$ obeys the logistic equation $\dot{x} = (a - b)x(1 - x)$. Explain why this implies that $x(t)$ increases monotonically and approaches 1 as $t \to \infty$.

Solution

Part (a)

Solve the initial value problems for $X(t)$ and $Y(t)$.

$$\frac{dX}{dt} = aX, \quad X(0) = X_0 \qquad\qquad \frac{dY}{dt} = bY, \quad Y(0) = Y_0$$

$$\frac{\frac{dX}{dt}}{X} = a \qquad\qquad \frac{\frac{dY}{dt}}{Y} = b$$

$$\frac{d}{dt}\ln|X| = a \qquad\qquad \frac{d}{dt}\ln|Y| = b$$

$$\ln|X| = at + C_1 \qquad\qquad \ln|Y| = bt + C_2$$

$$|X| = e^{at+C_1} \qquad\qquad |Y| = e^{bt+C_2}$$

$$X(t) = \pm e^{C_1}e^{at} \qquad\qquad Y(t) = \pm e^{C_2}e^{bt}$$

$$X(t) = A_1 e^{at} \qquad\qquad Y(t) = A_2 e^{bt}$$

Apply the initial conditions to determine A_1 and A_2.

$$X_0 = A_1 \qquad\qquad Y_0 = A_2$$

Therefore,

$$X(t) = X_0 e^{at} \qquad\qquad Y(t) = Y_0 e^{bt}.$$

Let $x(t)$ be the population fraction of species X.

$$x(t) = \frac{X(t)}{X(t) + Y(t)} = \frac{X_0 e^{at}}{X_0 e^{at} + Y_0 e^{bt}} = \frac{X_0}{X_0 + Y_0 e^{bt} e^{-at}} = \frac{X_0}{X_0 + Y_0 e^{(b-a)t}}$$

Since $b - a < 0$, the denominator gets smaller and smaller as t increases, meaning that $x(t)$ increases monotonically. In fact, $x(t)$ goes from

$$\lim_{t\to 0} x(t) = \frac{X_0}{X_0 + Y_0} \quad \text{to} \quad \lim_{t\to\infty} x(t) = \frac{X_0}{X_0} = 1.$$

Part (b)

Begin with the population fraction of species X.

$$x(t) = \frac{X(t)}{X(t) + Y(t)}$$

Differentiate both sides with respect to t.

$$\begin{aligned}
\frac{d}{dt}[x(t)] &= \frac{d}{dt}\left[\frac{X(t)}{X(t) + Y(t)}\right] \\
\dot{x}(t) &= \frac{\left[\frac{d}{dt}X(t)\right][X(t) + Y(t)] - \left\{\frac{d}{dt}[X(t) + Y(t)]\right\}X(t)}{[X(t) + Y(t)]^2} \\
&= \frac{\dot{X}(t)[X(t) + Y(t)] - [\dot{X}(t) + \dot{Y}(t)]X(t)}{[X(t) + Y(t)]^2} \\
&= \frac{\dot{X}(t)Y(t) - \dot{Y}(t)X(t)}{[X(t) + Y(t)]^2} \\
&= \frac{[aX(t)]Y(t) - [bY(t)]X(t)}{[X(t) + Y(t)]^2} \\
&= \frac{(a-b)X(t)Y(t)}{[X(t) + Y(t)]^2} \\
&= (a-b)\frac{X(t)}{X(t) + Y(t)}\frac{Y(t)}{X(t) + Y(t)} \\
&= (a-b)\frac{X(t)}{X(t) + Y(t)}\left[1 - \frac{X(t)}{X(t) + Y(t)}\right] \\
&= (a-b)x(t)[1 - x(t)] \\
&= (a-b)x(1-x)
\end{aligned}$$

As $t \to \infty$, the population of species X reaches equilibrium, that is, $\dot{x} = 0$ as $t \to \infty$.

$$(a-b)x^*(1 - x^*) = 0$$

$$x^* = 0 \quad \text{or} \quad x^* = 1$$

Since $0 < x(0) < 1$ and $a - b > 0$, $\dot{x}$ is always positive, meaning x grows monotonically and approaches 1 as $t \to \infty$.